Flight Mechanics of the 24-Hour Satellite

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The noncircularity of the parallels of latitude, including the equator, will cause the 24-hr equatorial satellite to drift in longitude. This paper presents simple expressions for the magnitude, period, and other parameters of these oscillations. The results show excellent agreement with those obtained by the tedious detailed machine integration of the classical differential equations of motion. Simple closed-form equations and dimensionless plots are presented for angular drift rate, altitude, velocity, and period of the satellite's mean path. For an equatorial ellipticity of 3.21 × 10⁻⁵, the maximum change in altitude of the mean path for one day is ±1275 ft. This occurs when the satellite is half-way between the major and minor axis of the equator. A diurnal oscillation can exist about this mean path. The maximum magnitude of the oscillation in this example is ±203 ft, which is 1/2π as great as the maximum daily change in altitude.

Nomenclature

A = maximum value of tangential component of gravity acceleration
g = gravity acceleration
r = radius distance from earth center to satellite
R = earth equatorial radius
t = time
V = velocity of satellite
δr = magnitude of daily oscillation in r from mean path
Δ = an increment or change in, generally the deviation from the 24-hr circular orbit condition
$\triangleq$ = equals by definition
ϵ = ellipticity of the equator
μ = gravity constant (distance^3/time^2) = r^2g
λ = satellite longitude measured from equatorial minor axis
τ = period of mean satellite drift
ω = earth rotational angle

Subscripts

M = at maximum positive or negative λ
0 = 24-hr circular orbital conditions

Blitzer[1] has shown that the rate of longitudinal drift of the 24-hr equatorial satellite due to equatorial oblateness is sufficiently large to be of concern to system designers. With the exception of the period for the case of small amplitude oscillation, Blitzer's work[1] did not present any closed-form analytical expressions for the mean path motion of the satellite. Such expressions, presented herein, are of value in developing station-keeping techniques.

The 24-hr satellite in the equatorial plane at a longitude between the minor and major equatorial axes is illustrated in Fig. 1. Because of the equatorial oblateness, the gravity vector is deflected from the center of the earth toward the major axis, as shown. This gravitational attraction may be divided into two components: one along the earth radius vector toward the center of rotation (center of the earth), referred to as the vertical component, and the other normal to the radius vector, referred to as the horizontal component. This latter component is very minute and has the same effect as a small thrust or drag. Previous studies by the author[2, 3] show that a small drag or a small thrust causes a sine wave oscillation in both altitude and velocity in which the mean velocity and altitude (achieved every half revolution) maintain the circular relationship

$$V^2 r = \mu \tag{1}$$

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In the case of drag[2] the altitude decreases, and in the case of forward thrust[3] the altitude increases. Both studies[2, 3] show that the rate of change of the *mean* tangential velocity is proportional to the tangential force and acts in the direction opposite to it. The tesseral harmonic of the classical gravity potential function[4] indicates that the tangential component of gravity caused by the equatorial ellipticity is proportional to the sine of twice the longitudinal angle measured from the minor equatorial axis. Therefore, the mean tangential satellite acceleration directed toward the *minor* axis is

$$\text{mean } \dot{V} = A \sin 2\lambda \tag{2}$$

where

$$A = \mu R^2 \epsilon / r^4 \tag{3}$$

Using observed satellite orbit perturbations as bases, Izsak[5] calculated $\epsilon = 3.21 \times 10^{-5}$ with the major axis at 33°W longitude. This value of ϵ gives the following value of A at the 24-hr circular orbital altitude:

$$A = 5.40 \times 10^{-7} \text{ ft/sec}^2 \tag{4}$$

As shown later, the satellite remains close to the 24-hr orbital altitude, and so the value A may be treated as a constant throughout the analysis. Obviously, this tangential component of acceleration is very minute compared to the total gravity acceleration in the 24-hr orbit (about 0.736 ft/sec^2), and the entire analysis may be treated using the perturbation techniques discussed in Refs. 2 and 3.

The general equation for radial acceleration without thrust is

$$\ddot{r} = (V^2/r) - (\dot{r}^2/r) - (\mu/r^2) \tag{5}$$

where the first two terms on the right-hand side are the centrifugal acceleration, and the last term is the inverse square gravity acceleration. In the case of a satellite starting with circular velocity at or near the 24-hr circular altitude, the vertical component of velocity $\dot{r}$ is comparatively low, and the equation may be reduced to the following:

$$\ddot{r} = (V^2/r) - (\mu/r^2) \tag{6}$$

Where the overall changes are small, Eq. (6) may be put into the following incremental form by omitting higher-order terms:

$$\Delta\ddot{r} = \frac{V_0^2}{r_0}\left(1 + 2\,\frac{\Delta V}{V_0} - \frac{\Delta r}{r_0}\right) - \frac{\mu}{r_0^2}\left(1 - 2\,\frac{\Delta r}{r_0}\right) \tag{7}$$

where

$$\Delta V/V_0 \ll 1.0 \qquad \Delta r/r_0 \ll 1.0 \tag{8}$$

For convenience, the subscript zero will be used to denote the 24-hr circular orbital conditions. Since the ideal 24-hr satellite rotational velocity V_0/r_0 is the same as the earth's rotational velocity $\dot{\omega}$, Eq. (7) can be reduced to the following by using the circular velocity-altitude relationship shown in Eq. (1):

$$\Delta\ddot{r} = \dot{\omega}(2\ \Delta V + \dot{\omega}\Delta r) \tag{9}$$

The total acceleration along the flight path is the sum of the effects of the vertical and horizontal components of gravity. Since the flight path angle is close to zero, its cosine may be treated as unity.

Then

$$\dot{V} = A\ \sin 2\lambda - (\mu/r^2)(\dot{r}/V) \tag{10}$$

Since the longitudinal drift rate $\dot{\lambda}$ is very small, λ in Eq. (10) may be treated as being constant for one or two revolutions about the earth. Equation (10) then may be written in incremental form [Eq. (11)] and then substituted into Eq. (9) to yield Eq. (12):

$$\Delta V = A\ \sin 2\lambda\ \Delta t - \dot{\omega}\Delta r \tag{11}$$

$$\Delta\ddot{r} = \dot{\omega}(2A\ \sin 2\lambda\ \Delta t - \dot{\omega}\Delta r) \tag{12}$$

With λ still being treated as a constant, Eq. (12) may be solved by a Laplace transformation to yield

$$\Delta r = (2A\ \sin 2\lambda/\dot{\omega}^2)(\Delta\omega - \sin\Delta\omega) \tag{13}$$

Equation (13) applies only over one or a relatively few revolutions of the earth, during which the satellite longitude is relatively unchanged. It does show that the increment in radius distance above or below the point at which the few revolutions start consists of two components: one proportional to the earth's rotational angle, and the other a sine wave oscillation of the same angle. The period of this sine wave oscillation is about one revolution, or one day. The magnitude of the oscillation is shown from Eq. (13) to be

$$\delta r = 2A\ \sin 2\lambda/\dot{\omega}^2 \tag{14}$$

Measuring A at the 24-hr orbit and substituting it from Eq. (3) into Eq. (14) shows δr to be proportional to the earth's equatorial ellipticity and to have its maximum value at a longitude of 45° from the minor axis:

$$\delta r = 2(R^2/r_0)\epsilon\ \sin 2\lambda \tag{15}$$

For $\epsilon = 3.21 \times 10^{-5}$, the maximum deviation δr from the mean path is 203 ft. Equation (13) shows that the corresponding maximum daily change in altitude is 1275 ft.

The existence of this diurnal oscillation (±203 ft maximum) is predicated upon the assumption that the trajectory starts in horizontal flight, $\dot{r}$ equal to zero, with $\ddot{r}$ also equal to zero. It can be shown that, if the vehicle is started originally down the mean flight path, these oscillations will not develop.

The velocity equation corresponding to Eq. (13) may be found by substituting Eq. (13) into Eq. (11):

$$\Delta V = (-A\ \sin 2\lambda/\dot{\omega})(\Delta\omega - 2\ \sin\Delta\omega) \tag{16}$$

Omitting the small oscillation from Eq. (13) results in an equation for the mean path which is applicable within the validity of a constant longitude λ:

$$\Delta r = (2A\ \sin 2\lambda/\dot{\omega}^2)\Delta\omega \tag{17}$$

The true differential equation for the mean path with a variable λ now may be found by letting the increments of Eq. (17) diminish to infinitesimals and by substituting $\dot{\omega}d\lambda/\dot{\lambda}$ for $d\omega$:

$$dr = (2A\ \sin 2\lambda/\dot{\omega})(d\lambda/\dot{\lambda}) \tag{18}$$

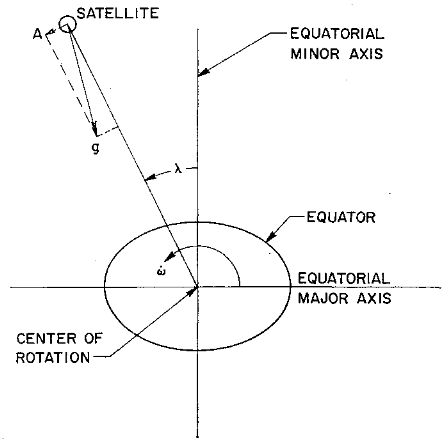

Fig. 1 Illustration of tangential gravity component

Before integrating Eq. (18), it is necessary to find a suitable expression for $\dot{\lambda}$. The longitudinal drift rate is equal to the difference between the satellite's angular velocity and the earth's rotational velocity:

$$\dot{\lambda} = (V/r) - (V_0/r_0) \tag{19}$$

Equation (19) now may be put in incremental form:

$$\dot{\lambda} = \dot{\omega}[(\Delta V/V_0) - (\Delta r/r_0)] \tag{20}$$

The circular velocity relationship, $V^2r = \mu$ [Eq. (1)], which in effect states that the centrifugal acceleration is balanced by the gravity acceleration, next is put in incremental form for the 24-hr orbit:

$$\Delta r/r_0 = -2(\Delta V/V_0) \tag{21}$$

Substituting Eq. (21) into Eq. (20) gives

$$\dot{\lambda} = 3\dot{\omega}(\Delta V/V_0) = 3(\Delta V/r_0) \tag{22}$$

Differentiating Eq. (22) yields

$$\ddot{} = (3/r_0)\dot{V} \tag{23}$$

Similarly to the derivation of Eq. (17) for Δr, the omission of the oscillation in $\Delta\omega$ from Eq. (16) results in an equation for the mean path:

$$\Delta V = -(A\ \sin 2\lambda/\dot{\omega})\Delta\omega \tag{24}$$

Equation (24) is valid only for $\sin 2\lambda$ remaining virtually unchanged. Differentiation of Eq. (24), however, results in a true differential equation, applicable along the mean path, in which the drift angle λ may be treated as a variable:

$$\dot{V} = -A\ \sin 2\lambda \tag{25}$$

Substituting Eq. (25) into Eq. (23) yields

$$\ddot{} = (-3A/r_0)\ \sin 2\lambda \tag{26}$$

the solution of which is the final equation for the angular drift rate

$$\dot{\lambda} = [(3A/r_0)(\cos 2\lambda - \cos 2\lambda_M)]^{1/2} \tag{27}$$

Since the radical can be either positive or negative, it is obvious that the drift rate is symmetrical about the minor equatorial axis from which the drift angle is measured. To obtain the equation for the change in altitude from the circular 24-hr altitude, the next step is to substitute $\dot{\lambda}$ from Eq. (27) into Eq. (18) and to integrate the left-hand side of the equation:

$$\Delta r = \frac{-A}{\dot{\omega}}\left(\frac{r_0}{3A}\right)^{1/2}\int_{2\lambda_M}^{2\lambda}\frac{d\cos 2\lambda}{(\cos 2\lambda - \cos 2\lambda_M)^{1/2}} \tag{28}$$

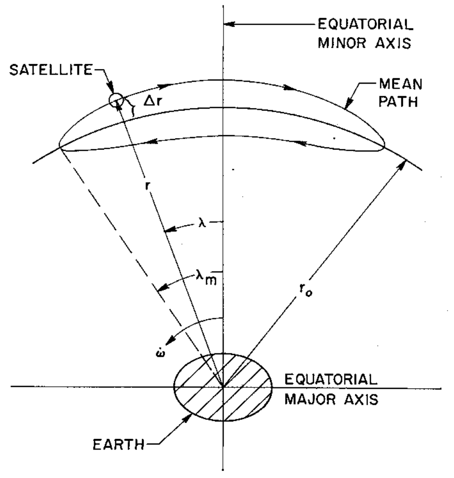

Fig. 2 Geometry of satellite mean path

or

$$\Delta r = (2/\dot{\omega})(Ar_0/3)^{1/2}(\cos 2\lambda - \cos 2\lambda_M)^{1/2} \tag{29}$$

The presence of the radical and the form of Eq. (29) show that the deviation in altitude from the 24-hr circular altitude is symmetrical about both the 24-hr altitude and the equatorial minor axis. Equation (26) shows that the angular acceleration is always toward the minor axis. Equation (21) indicates that an increase in velocity results in a decrease in radius distance, and vice versa. Therefore, as illustrated in Fig. 2, the drift of the satellite is in the direction of earth rotation when the altitude is below the 24-hr value and is in the opposite direction when the altitude is high.

The maximum deviation in radius distance from the circular altitude occurs as the satellite crosses the minor axis and is found from Eq. (29) to be

$$\Delta r_{\lambda=0} = \pm(2/\dot{\omega})(2r_0A/3)^{1/2}\sin\lambda_M$$
$$= \pm 2R(\tfrac{2}{3}\epsilon)^{1/2}\sin\lambda_M \tag{30}$$

Setting the sine of λ equal to unity in Eq. (30) and using 5.40×10^{-7} ft/sec² for A shows the maximum fractional deviation distance from the center of the earth to be 1.4×10^{-3}. Equation (3) indicates that A is inversely proportional to the fourth power of r. Therefore, the maximum fractional deviation in A from the 24-hr value is less than 6×10^{-3}, which justifies treating A as a constant throughout the analysis. The principal equations of this analysis indicate that it is the one half power of A that influences the orbital parameters. The maximum fractional deviation in $A^{1/2}$ is 0.28%, and since it is the integrated average effect that counts, it is safe to say that the maximum fractional error in the incremental orbital parameters is not more than half of this value, or 0.15%. If greater accuracy is required, it can be shown that virtually all of the error in ΔV and Δr may be removed by multiplying them by $1 - \Delta r/r_0$, which is the same as $1 + 2(\Delta V/V_0)$. It also can be shown that the finite error in the quarter period expressed in days is $\pm 2\lambda_M/3\pi$, where λ_M is expressed in radians. The positive sign should be used when the altitude is above the 24-hr circular value and the negative sign when the altitude is below this value. Since the drift trajectory is symmetrical, the time errors should balance out, and since A has been treated as a constant, the total period should not require correction.

If the deviation in altitude from the circular orbit exceeds the maximum value attainable by Eq. (30), the drift velocity will be finite as the vehicle reaches the major equatorial axis, and the vehicle then will cross this axis. Once this is

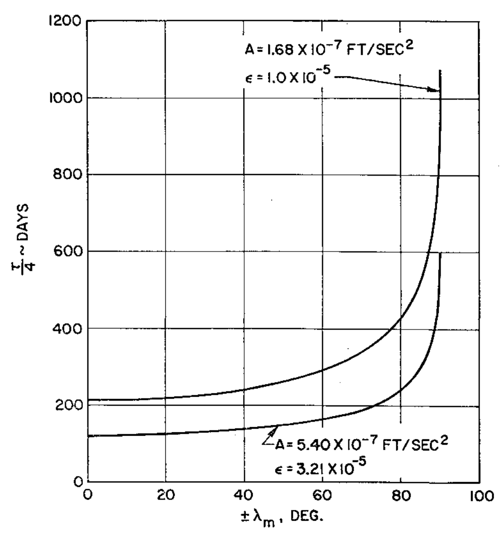

Fig. 3 Satellite quarter period for two values of maximum horizontal gravity acceleration

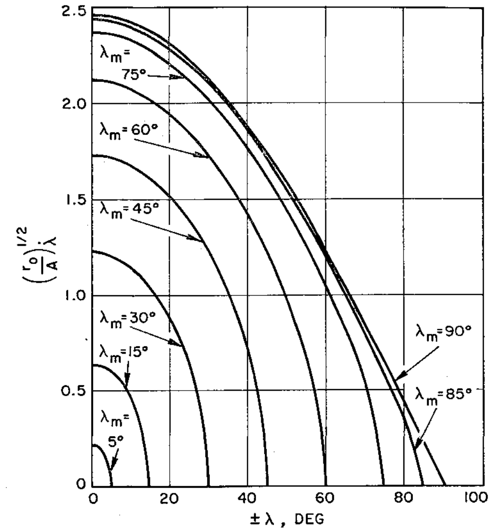

Fig. 4. Angular drift rate as a function of drift angle

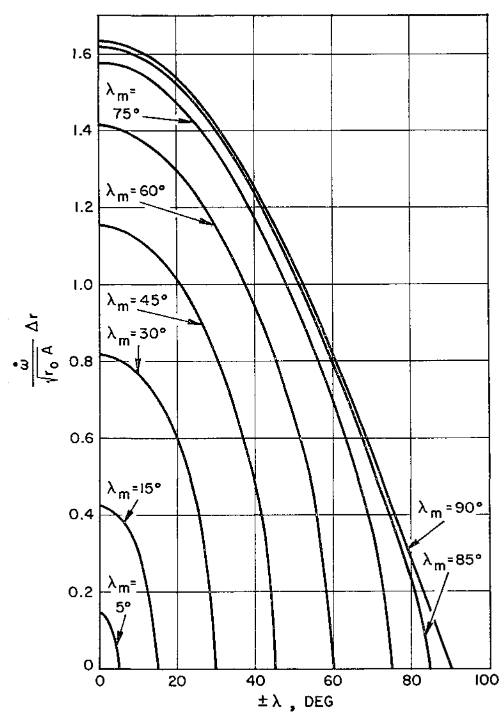

Fig. 5 Incremental radius as a function of drift angle

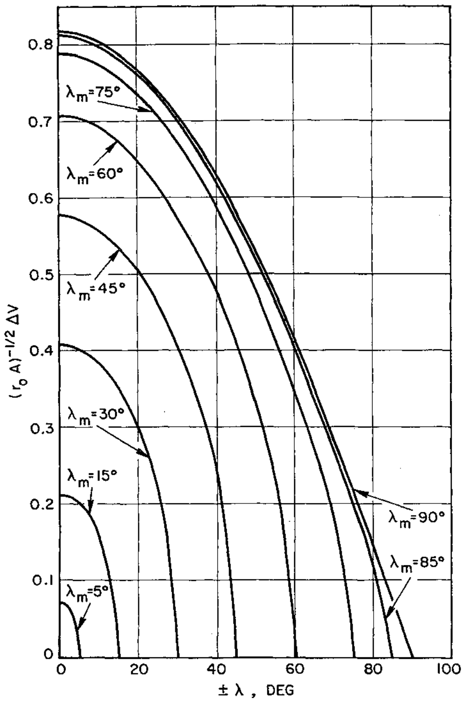

Fig. 6 Incremental velocity as a function of drift angle

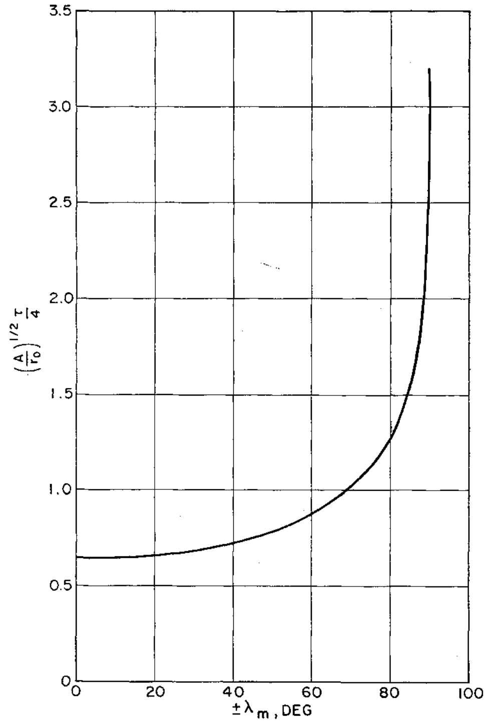

Fig. 7 Quarter period as a function of maximum drift angle

accomplished, the direction of the tangential gravity acceleration A is reversed, and the satellite will continue to drift all the way around the earth repeating the same mean path. The analytical expressions derived herein for the satellite mean path are limited to a maximum drift angle λ_M of 90°.

Substitution of Δr from Eq. (29) into Eq. (21) results in the final equation for the drift velocity along the mean path:

$$\Delta V = (Ar_0/3)^{1/2}(\cos 2\lambda - \cos 2\lambda_M)^{1/2} \tag{31}$$

The period of the mean satellite drift about the minor axis may be found by substituting $\dot{\lambda}$ from Eq. (27) into the following and integrating:

$$dt = d\lambda/\dot{\lambda} \tag{32}$$

Because the motion is symmetrical (Fig. 2), the quarter period may be obtained by integrating between $\lambda = 0$ and λ_M:

$$\frac{\tau}{4} = \left(\frac{r_0}{3A}\right)^{1/2} \int_0^{\lambda_M} \frac{d\lambda}{(\cos 2\lambda - \cos 2\lambda_M)^{1/2}} \tag{33}$$

Equation (33) is an elliptic integral of the first kind which easily is integrated after being put into the following form:

$$\frac{\tau}{4} = \left(\frac{r_0}{6A}\right)^{1/2} \int_0^{\pi/2} \frac{d\phi}{(1 - \sin^2\lambda_M \sin^2\phi)^{1/2}} \tag{34}$$

where

$$\sin\phi \triangleq \sin\lambda/\sin\lambda_M \tag{35}$$

Figure 3 is a plot of the quarter period as a function of the maximum drift angle λ_M for two arbitrary values of the maximum horizontal component of gravity acceleration A. When the amplitude of the oscillation λ_M is set at zero in Eq. (34), the value of the integral is $\pi/2$, and the equation reduces to Blitzer's equation[1] for the period of a low-amplitude oscillation.

Substitution of A from Eq. (3) into Eqs. (27, 29, and 31) shows that $\dot{\lambda}$, Δr, and ΔV, respectively, along the mean path are all proportional to the square root of the earth's equatorial ellipticity ϵ, and Eq. (34) shows the period to be inversely proportional to the square root of ϵ. On the other hand, the minute diurnal oscillations are seen from Eq. (13) or (15) and from Eq. (16) to be proportional to the first power of ϵ.

Since the maximum value of the horizontal component of gravity acceleration near the 24-hr orbit presently is not known, the principal equations for the mean satellite motion are plotted in dimensionless form. Equations (27, 29, 31, and 34) for $\dot{\lambda}$, Δr, ΔV, and $\tau/4$, respectively, are plotted in Figs. 4–7.

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[2] Perkins, F. M., "An analytical solution for flight time of satellites in eccentric and circular orbits," Astronaut. Acta **11**, 117–119 (1958).

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